Int. J. Solids Structures, 1976, Vol. 12, pp. 467–477. Pergamon Press. Printed in Great Britain

NONLINEAR WAVES GENERATED BY A STEADILY MOVING LINE LOAD ON AN ELASTIC PLATE†

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(*Received* 31 *December* 1973; *revised* 17 *November* 1975)

Abstract—The steady-state response of an infinite plate to a steadily moving line load is studied. The nonlinear plate theory of Herrmann is used. The plate response is governed by a set of nonlinear differential equations and, in addition, must satisfy the "radiation" conditions. Appropriate radiation conditions for the present nonlinear problem are developed. Exact solutions representing nonlinear waves generated by the moving load are constructed.

1. INTRODUCTION

The dynamic response of structural elements such as elastic beams and plates to moving loads has been the subject of numerous studies in the engineering literature using various linear theories. It is not the purpose of the present work to review such studies. We wish to point out, however, that the validity of such linear solutions is generally limited to small load intensities and to loads with speeds that are away from certain "critical speeds." When such limitations are violated, solutions on the basis of nonlinear theories are then called for. As nonlinear problems are intrinsically more difficult to solve, there have been but a few available nonlinear studies on the subject. In this connection we mention the recent publications [1–3], wherein attempts were made to obtain nonlinear beam solutions using the perturbation method.

We consider here an infinite, elastic plate under the action of a steadily moving line load. We shall use the nonlinear plate theory of Herrmann [4], which is a dynamic version of the von Karman plate theory [5]. We are interested in the steady-state response of the plate representing waves generated by the moving load and shall construct exact nonlinear solutions as a basis for assessing the validity of the linear solutions as well as the nonlinear perturbation solutions.

The mathematical problem, as formulated in Section 2 in a moving reference frame, reduces to that of solving a nonlinear, second order ordinary differential equation of the Duffing type for the dependent variable φ, which is the derivative of the transverse plate displacement with respect to the moving variable (see eqn 6). We note that the same equation but without the loading term was also previously derived by Advani [6]. He showed the existence of a class of exact travelling wave solutions for the plate in the absence of external loads and inquired as to how such waves might be established. Our solutions thus provide some partial answer to Advani's inquiry.

The nonlinear equation for φ as mentioned above has a two-parameter family of solutions. One thus has the difficult task of selecting from such solutions the unique one that represents waves emanating from the load. For linear problems such "outgoing wave" or "radiation" conditions are well known and easy to pose, but there does not seem to exist any well-established principle for posing such radiation conditions for nonlinear problems. We develop such conditions here by transforming the nonlinear problem into an infinite set of linear problems through a perturbation expansion and applying the radiation conditions to the linear problems. However, we do not actually construct the formal series nor prove its convergence. By examining the manner in which the terms in the series are obtained, we are able to deduce the appropriate conditions which the nonlinear solution must satisfy if the formal series does converge. With such radiation

†Presented at the Seventh U. S. National Congress of Applied Mechanics, University of Colorado, Boulder, Colorado, 3–7 June, 1974. The research was supported by the U.S. National Science Foundation under Grant No. GK-35162.

conditions known we then proceed directly to construct the exact solution of the nonlinear problem.

The linear problem is of two different types, depending on whether $p^2>0$ or $p^2<0$ where p^2 is a parameter depending on the load speed. In Sections 3 and 4 the cases $p^2>0$ and $p^2<0$ are studied separately. For $p^2>0$, a unique nonlinear solution is obtained that is valid for all values of the load intensity. For $p^2<0$, however, the nonlinear solution so constructed exists only for a finite range of values of the load intensity. For large values of the load intensity a new type of solution similar to that for $p^2>0$ is found to exist. This latter solution satisfies a different set of radiation conditions which are discussed in Section 4.

In Section 5 we consider the case $p^2=0$ and discuss the physical meaning of p^2. Other discussions and remarks are given in Section 6.

2. FORMULATION

Let (x, y, z) denote Cartesian coordinates and (u, v, w) the corresponding displacement vector. We choose the undeformed midplane of the plate to coincide with the x, y plane and set, as usual,

$$u(x, y, z, t) = u_0(x, y, t) - z\frac{\partial w_0}{\partial x} \tag{1a}$$

$$v(x, y, z, t) = v_0(x, y, t) - z\frac{\partial w_0}{\partial y} \tag{1b}$$

$$w(x, y, z, t) = w_0(x, y, t). \tag{1c}$$

We take the transverse load q in the form of a line load that extends in the y-direction and has a constant speed c in the x-direction

$$q = A\delta(x - ct) \tag{2}$$

where A is the load intensity and δ denotes the Dirac function. We then seek "steady-state" solutions, i.e. solutions that are functions of

$$\tau \equiv x - ct \tag{3}$$

only. Following[4], the governing equations in terms of u_0, v_0 and w_0 may be written as

$$\left[(c_p^2 - c^2)u_0' + \frac{1}{2}c_p^2(w_0')^2\right]' = 0 \tag{4a}$$

$$(c_s^2 - c^2)v_0'' = 0 \tag{4b}$$

$$\rho h\left[-\frac{h^2}{12}(c_p^2 - c^2)w_0''' + c_p^2\left(u_0'w_0' + \frac{1}{2}(w_0')^3\right) - c^2 w_0'\right]' = -A\delta(\tau) \tag{4c}$$

where $c_p = [E/\rho(1-\nu^2)]^{1/2}$ is the flexural wave speed, $c_s = (G/\rho)^{1/2}$ is the shear wave speed, with E being Young's modulus, G being the shear modulus and υ being Poisson's ratio, ρ is the density, and h is the plate thickness. The primes in (4) signify differentiations with respect to τ.

It is seen from (4b) above that the equation for v_0 is uncoupled from those for u_0 and w_0. As eqn (4b) does not involve the applied load we shall not consider it any further.

The equations for u_0 and w_0 as given in (4a) and (4c) are coupled. Dividing (4a) by c_p^2 and (4c) by $\rho h c_p^2$ and integrating the resulting equations with respect to τ, we obtain

$$(1 - \lambda^2)u_0' + \frac{1}{2}(w_0')^2 = D_1 \tag{5a}$$

$$-\frac{h^2}{12}(1 - \lambda^2)w_0''' + u_0'w_0' + \frac{1}{2}(w_0')^3 - \lambda^2 w_0' = -\frac{A}{\rho h c_p^2}H(\tau) + D_2 \tag{5b}$$

where $\lambda = c/c_p$, D_1 and D_2 are constants of integration and H denotes the Heaviside function. We take $D_2 = A/\rho h c_p^2$ so that the right-hand side of (5b) vanishes identically ahead of the load ($\tau > 0$). Eliminating u_0' from (5a) and (5b) and introducing $\varphi = w_0'$, we finally obtain

$$\varphi'' + p^2(\varphi + \beta\varphi^3) = -BH(-\tau) \tag{6}$$

where

$$p^2 = 12(\lambda^2 - \lambda^4 - D_1)/h^2(1-\lambda^2)^2 \tag{7a}$$

$$\beta = \lambda^2/2(\lambda^2 - \lambda^4 - D_1) \tag{7b}$$

$$B = 12A/\rho h^3 c_p^2(1-\lambda^2). \tag{7c}$$

As was mentioned in the Introduction, the above Duffing-type equation, with the right-hand side being zero, was also given in[6].

We now consider eqn (6) for φ, regarding B and λ as given. It can be shown that, for arbitrarily chosen $\varphi(0)$ and $\varphi'(0)$, eqn (6) has unique solutions for both $\tau > 0$ and $\tau < 0$. So there is a two-parameter family of solutions for φ. We shall develop radiation conditions below that enable us to determine those solutions of (6) that represent outgoing waves.

3. THE CASE $p^2 > 0$

We shall assume $c \neq c_p$ so that $\lambda \neq 1$. In order to derive radiation conditions for eqn (6) we shall transform it into a system of linear equations. This is achieved by expanding φ in a formal series in B. The leading member in the system corresponds to the linearized form of eqn (6), whose solution takes different form for $p^2 > 0$ and $p^2 < 0$. Since the linear solution is used to generate solutions of the succeeding members in the system, we thus consider the cases $p^2 > 0$ and $p^2 < 0$ separately in this and the next section. The case $p^2 = 0$ will be discussed in Section 5.

$p^2 > 0$ implies $\beta > 0$. For convenience we introduce $p_1^2 = p^2 > 0$, $\beta_1 = \beta > 0$ as defined in (7a) and (7b) and rewrite eqn (6) as

$$\varphi'' + p_1^2(\varphi + \beta_1\varphi^3) = -BH(-\tau). \tag{8}$$

We now regard B as a small parameter and expand φ as a power series in B. We expect φ to vanish identically when $B = 0$ and to be an odd function of B. Setting $B = \epsilon$ we now write

$$\varphi = \sum_{i=0}^{\infty} \varphi_i \epsilon^{i+1}, \qquad i = \text{even}. \tag{9}$$

Substituting (9) into (8) and collecting like powers of ϵ yields the system of linear equations

$$\varphi_0'' + p_1^2\varphi_0 = -H(-\tau) \tag{10a}$$

$$\varphi_2'' + p_1^2\varphi_2 = -p_1^2\beta_1\varphi_0^3 \tag{10b}$$

$$\varphi_4'' + p_1^2\varphi_4 = -3p_1^2\beta_1(\varphi_0^2\varphi_2 + \varphi_0\varphi_2^2). \tag{10c}$$

We consider the solution of these equations below.

We must develop conditions so that the solutions for the φ_i's in eqn (10) are unique and represent waves that are generated by the applied load. Alternatively, these same conditions must render the solutions of the homogeneous equations associated with (10) identically zero, for then there is no load on the plate and the solutions would represent waves that are generated at infinity. The homogeneous equations associated with (10) have the general solution $\varphi_h(\tau)$ consisting of the linear combinations of $\exp(\pm ip_1\tau)$ which are bounded for all τ. The usual boundedness requirement on the φ_i's is thus insufficient to make $\varphi_h(\tau)$ vanish. This difficulty may be resolved by introducing a small damping, i.e. by adding a term $c\varphi'$, $c > 0$, into eqn (10), requiring the solutions to tend to zero as $\tau \to \pm\infty$, and then taking their limits as $c \to 0$. One can easily verify that the solutions $\exp(\pm ip_1\tau)$ then go to $\exp(q \pm ip_1\tau)$, where q is real and negative and tends to zero as $\tau \to 0$. Thus the solutions $\exp(\pm ip_1\tau)$ are allowed for $\tau > 0$ but not for $\tau < 0$. The vanishing of $\varphi_h(\tau)$ for $\tau < 0$ together with the continuity of both $\varphi_h(\tau)$ and $\varphi_h'(\tau)$ at $\tau = 0$ then renders $\varphi_h(\tau)$ identically zero.

We now write the solution for φ_0 in (10a) as

$$\varphi_0(\tau) = \begin{cases} A_0^+ \exp(ip_1\tau) + B_0^+ \exp(-ip_1\tau) & \tau > 0 \\ -1/p_1^2 & \tau < 0 \end{cases} \quad (11)$$

Applying the condition of continuity of $\varphi_0(\tau)$ and $\varphi_0'(\tau)$ at $\tau = 0$, we determine A_0^+ and B_0^+ and obtain

$$\varphi_0(\tau) = \begin{cases} (-1/p_1^2)\cos p_1\tau & \tau > 0 \\ -1/p_1^2 & \tau < 0 \end{cases} \quad (12)$$

We remark that $B\phi_0(\tau)$ is the linearized solution of eqn (8). The graph for $B\varphi_0(\tau)$ is shown in Fig. 1.

We determine $\varphi_2, \varphi_4, \ldots$ in similar manner. We write the solution for φ_2 in (10b) as

$$\varphi_2(\tau) = \begin{cases} A_2^+ \exp(ip_1\tau) + B_2^+ \exp(-ip_1\tau) + \phi_{2p}^+ & \tau > 0 \\ \varphi_{2p}^- & \tau < 0 \end{cases} \quad (13)$$

where φ_{2p}^+ and φ_{2p}^- are particular solutions of (10b) for $\tau > 0$ and $\tau < 0$ respectively. The constants A_2^+ and B_2^+ are again determined by the continuity of $\varphi_2(\tau)$ and $\varphi_2'(\tau)$ at $\tau = 0$.

We now make an important observation: As φ_0 is constant for $\tau < 0$ and the right-hand side of (10b) depends only on φ_0, the right-hand side.of (10b) is constant for $\tau < 0$. This in turn implies that φ_{2p}^- is a constant and φ_2 is constant for $\tau = 0$. By a simple induction it follows that $\varphi_4, \ldots$ are all constant for $\tau < 0$. *Thus the nonlinear solution φ as represented by the formal series in* (9) *is constant behind the load* ($\tau < 0$).

We now return to solve the nonlinear eqn (8) under the condition that φ be constant for $\tau < 0$. Consider first the tail wave, i.e. the solution for $\tau < 0$. As $\varphi' = \varphi'' = 0$ for $\tau < 0$, the tail wave is a constant solution of the algebraic equation

$$p_1^2(\varphi + \beta_1\varphi^3) = -B \quad (14)$$

It can be shown [7] that the above cubic equation has only one real root, F_1 say, which is given by

$$F_1 = (1/3\beta_1)^{1/2}\{[-\gamma_1 + (1+\gamma_1^2)^{1/2}]^{1/3} + [-\gamma_1 - (1+\gamma_1^2)^{1/2}]^{1/3}\} \quad (15)$$

where

$$\gamma_1 = \frac{3B}{2p_1^2}(3\beta_1)^{1/2} \quad (16)$$

Thus

$$\varphi(\tau) = F_1 \quad \tau < 0 \quad (17)$$

The solution for the head wave, i.e. for $\tau > 0$, may be obtained by considering eqn (8) in the phase plane. With the known conditions $\varphi(0) = F_1$ and $\varphi'(0) = 0$, the solution for the head wave may be

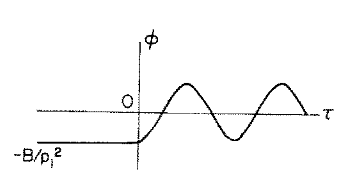

Fig. 1. Linear solution for $p^2 < 0$.

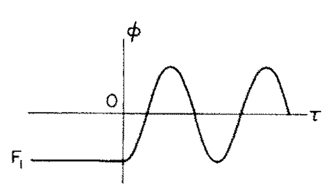

Fig. 2. Nonlinear solution for $p^2 < 0$.

expressed in terms of the Jacobian elliptic function[8]

$$\varphi(\tau) = F_1 cn\{(1+\delta_1^2)^{1/2} p_1\tau, \delta_1/[2(1+\delta_1^2)]^{1/2}\} \quad \tau > 0 \tag{18}$$

where $\delta_1^2 \equiv \beta_1 F_1^2$. A graph for $\varphi(\tau)$ is shown in Fig. 2.

We remark that the condition φ = constant for $\tau < 0$ which led to the nonlinear solution $\varphi(\tau)$ given in eqns (17) and (18) may also be derived in a more direct, but perhaps less rigorous, manner as follows: The moving line load on the plate is given as a delta function in eqn (2), whose integral as appearing in eqn (6) is taken as a step pressure. By letting $\tau \to -\infty$ we are considering a region of the plate where the pressure front has long past, and we expect φ to approach a static solution of eqn (8) under a constant pressure. It is easily seen that this static solution must be F_1. Now, on multiplying eqn (8) by φ', integrating once for $\tau < 0$, and examining the resulting expression, making use of $\varphi(-\infty) = F_1$ and $\varphi'(-\infty) = 0$, one can show that $\varphi'(\tau)$ must vanish identically and hence $\varphi \equiv F_1$ for $\tau < 0$. The details, however, will be omitted here.

4. THE CASE $p^2 < 0$

When $p^2 < 0$, $\beta < 0$. We now introduce $p_2^2 = -p^2 > 0$, $\beta_2 = -\beta > 0$ and rewrite eqn (6) as

$$\varphi'' - p_2^2(\varphi - \beta_2\varphi^3) = -BH(-\tau). \tag{19}$$

The same expansion for φ as given in eqn (9) now leads to the following system

$$\varphi_0'' - p_2^2\varphi_0 = -H(-\tau) \tag{20a}$$
$$\varphi_2'' - p_2^2\varphi_2 = -p_2^2\beta_2\varphi_0^3 \tag{20b}$$
$$\varphi_4'' - p_2^2\varphi_4 = -3p_2^2\beta_2(\varphi_0^2\varphi_2 + \varphi_0\varphi_2^2). \tag{20c}$$

The homogeneous equations associated with (20) have the general solution $\varphi_h(\tau)$ consisting of the linear combinations of $\exp(\pm p_2\tau)$. We now pose the condition that the φ_i's remain bounded as $\tau \to \pm\infty$. It is easy to see that this condition renders $\varphi_h(\tau)$ identically zero.

We now write the solution for φ_0 in (20a) as

$$\varphi_0(\tau) = \begin{cases} A_0 \exp(-p_2\tau) & \tau > 0 \\ B_0 \exp(p_2\tau) + (1/p_2^2) & \tau < 0 \end{cases} \tag{21}$$

which, after the constants A_0 and B_0 are determined by the continuity of $\varphi_0(\tau)$ and $\varphi_0'(\tau)$ at $\tau = 0$, becomes

$$\varphi_0(\tau) = \begin{cases} (1/2p_2^2)\exp(-p_2\tau) & \tau > 0 \\ (1/2p_2^2)\exp(p_2\tau) + (1/p_2^2) & \tau < 0 \end{cases}. \tag{22}$$

The graph of $B\varphi_0(\tau)$, which is the linearized solution of eqn (19), is shown in Fig. 3.

Similarly we write the solution for φ_2 in (20b) as

$$\varphi_2(\tau) = \begin{cases} A_2 \exp(-p_2\tau) + \varphi_{2p}^+ \\ B_2 \exp(p_2\tau) + \varphi_{2p}^- \end{cases} \tag{23}$$

where φ_{2p}^+ and φ_{2p}^- are particular solutions of (20b) for $\tau > 0$ and $\tau < 0$ respectively and the constants A_2 and B_2 are determined by the continuity of $\varphi_2(\tau)$ and $\varphi_2'(\tau)$ at $\tau = 0$.

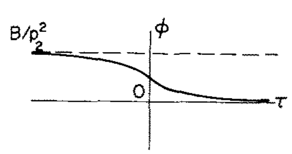

Fig. 3. Linear solution for $p^2 > 0$.

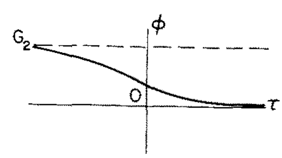

Fig. 4. Nonlinear solution for $p^2 > 0$, $B \leq B_{cr}$.

We observe that $\varphi_0(\tau)$ as given in (22) tends to zero exponentially as $\tau\to\infty$ and to a constant exponentially as $\tau\to-\infty$. This is also true of φ_{2p}^+ and φ_{2p}^- and hence of $\varphi_2(\tau)$ in (23). By a simple induction, considering the manner in which φ_{ip}^+ and φ_{ip}^+ are formed, $i=4,6,\ldots$, it follows that $\varphi_4(\tau)$, $\varphi_6(\tau),\ldots$ all enjoy the same property. *Thus the nonlinear solution $\varphi(\tau)$ of eqn* (19) *as represented by the formal series in eqn* (9) *tends to zero as $\tau\to\infty$ and to some constant as $\tau\to-\infty$, with $\varphi'(\tau)$, $\varphi''(\tau),\ldots$ all tending to zero as $\tau\to\pm\infty$.*

We can now solve eqn (19) under the further condition mentioned above. As $\tau\to-\infty$, $\varphi(\tau)\to G$ say and $\varphi''(\tau)\to 0$. Equation (19) then becomes

$$-p_2^2(G-\beta_2G^3)+B=0. \tag{24}$$

Following[7], the above cubic equation has for $B\le B_{cr}$,

$$B_{cr}\equiv(2/3)p_2^2(1/3\beta_2)^{1/2} \tag{25}$$

three real roots G_1, G_2 and G_3, with $G_1<G_2\le G_3$. We take

$$G=G_2=(1/3\beta_2)^{1/2}[\cos(\theta/3)-\sqrt{3}\sin(\theta/3)] \tag{26}$$

with

$$\theta=\tan^{-1}[(1-\gamma_2^2)^{1/2}/\gamma_2],\quad 0\le\theta\le\pi/2 \tag{27}$$

where $\gamma_2\equiv B/B_{cr}\le 1$. We take $G=G_2$ because $G_2=0$ when $B=0$.

We multiply eqn (19) by φ' and integrate it once to obtain

$$\frac{(\varphi')^2}{2}-\frac{p_2^2\varphi^2}{2}+\frac{p_2^2\beta_2\varphi^4}{4}=C_h,\quad \tau>0 \tag{28a}$$

$$\frac{(\varphi')^2}{2}-\frac{p_2^2\varphi^2}{2}+\frac{p_2^2\beta_2\varphi^4}{4}+B\varphi=C_t,\quad \tau<0. \tag{28b}$$

By the conditions $\varphi(\infty)=\varphi'(\infty)=\varphi'(-\infty)=0$ and $\varphi(-\infty)=G_2$ we can determine the constants C_h and C_t as

$$C_h=0,\ C_t=\frac{p_2^2\delta_2^2(2-\delta_2^2)}{12\beta_2} \tag{29}$$

where $\delta_2^2\equiv 3\beta_2G_2^2$. Now by the continuity of $\varphi(\tau)$ and $\varphi'(\tau)$ at $\tau=0$ we can determine $\varphi(0)$ and $\varphi'(0)$. Thus

$$\varphi(0)=\frac{C_t}{B}=\sqrt{\left(\frac{3}{\beta_2}\right)}\frac{\delta_2(2-\delta_2^2)}{4(3-\delta_2^2)} \tag{30}$$

and $\varphi'(0)$ can be expressed in terms of $\varphi(0)$ using either (28a) or (28b).

With C_h, C_t, $\varphi(0)$ and $\varphi'(0)$ known, we can determine $\varphi(\tau)$, for both $\tau>0$ and $\tau<0$, by solving for $\varphi'(\tau)$ in eqns (28a) and (28b) in terms of φ, integrating τ as a function of φ, and then inverting. Omitting the details, we have for the head wave

$$\varphi(\tau)=(2/\beta)^{1/2}\operatorname{sech}p_2(\tau+\tau_1),\quad \tau>0,\quad B\le B_{cr}. \tag{31}$$

The final expression for the tail wave for $B<B_{cr}$ is quite lengthy. We present it in the implicit form

$$\frac{4}{\beta_2}(1-\delta_2^2)+4\delta_2(1/3\beta_2)^{1/2}[\delta_2(1/3\beta_2)^{1/2}-\varphi]+2(2/\beta_2)^{1/2}(1-\delta_2^2)^{1/2}[(2-\delta_2^2)/\beta_2$$
$$-2\delta_2(1/3\beta_2)^{1/2}\varphi-\varphi^2]^{1/2}=[\delta_2(1/3\beta_2)^{1/2}-\varphi]\exp[(1-\delta_2^2)^{1/2}p_2(\tau-\tau_2),$$
$$\tau<0,\quad B<B_{cr} \tag{32}$$

though an explicit expression may be obtained by squaring eqn (32) above and then solving a quadratic equation for φ, For $B = B_{cr}$, we have

$$\varphi(\tau) = (1/3\beta_2)^{1/2} \frac{p_2^2(\tau - \tau_2)^2 - (9/2)}{p_2^2(\tau - \tau_2)^2 + (3/2)}, \quad \tau < 0, \quad B = B_{cr} \tag{33}$$

We remark that the constants τ_1 and τ_2 in eqns (31)–(33) above are to be determined so that eqn (30) holds. The graph for the $\varphi(\tau)$ obtained here is shown in Fig. 4.

When $B > B_{cr}$, the root G_2 of eqn (24) becomes complex and the solution $\varphi(\tau)$ given in eqns (31)–(33) ceases to exist. Equation (24) for $\gamma_2 \equiv B/B_{cr} > 1$ has only the real root $G = G_1$ given by

$$G_1 = (1/3\beta_2)^{1/2}\{[-\gamma_2 + (\gamma_2^2 - 1)^{1/2}]^{1/3} + [-\gamma_2 - (\gamma_2^2 - 1)^{1/2}]^{1/3}\}. \tag{34}$$

We can obtain a new type of solution of eqn (19) by requiring φ to approach a static solution of eqn (19) under a constant pressure as we discussed near the end of Section 3. Setting $\varphi(-\infty) = G_1$, which is the only static solution of (19) for $\gamma_2 > 1$, one can also show that $\varphi'(\tau)$ must then vanish identically for $\tau < 0$. Hence we have

$$\varphi(\tau) \equiv G_1 \quad \tau < 0. \tag{35}$$

By the continuity of $\varphi(\tau)$ and $\varphi'(\tau)$ at $\tau = 0$ we have from eqn (35) $\varphi(0) = G_1$ and $\varphi'(0) = 0$. The solution for the head wave is now uniquely determined and is obtained by integrating eqn (19) as before. The explicit solution for $\varphi(\tau)$, $\tau > 0$, takes different forms depending on the value of B. We have the following subcases:

(i) For $G_1 < -(2/\beta_2)^{1/2}$ or, equivalently, for $B < \gamma_2 B_{cr}$, where $\gamma_3 = 3.6742$ and is the root of the equation

$$-\sqrt{6} = [-\gamma + (\gamma^2 - 1)^{1/2}]^{1/3} + [-\gamma - (\gamma^2 - 1)^{1/2}]^{1/3} \tag{36}$$

(see eqn 34), we have

$$\varphi(\tau) = G_1 \,\mathrm{dn}\,\{\delta_3 p_2 \tau/\sqrt{2}, [2(\delta_3^2 - 1)]^{1/2}/\delta_3\} \quad \tau > 0 \tag{37}$$

where $\delta_3^3 \equiv \beta_2 G_1^2 < 2$. The Jacobian elliptic function dn is defined in [8].

(ii) For $G_1 = -(2/\beta_1)^{1/2}$, or $B = \gamma_3 B_{cr}$, we have

$$\varphi(\tau) = G_1 \,\mathrm{sech}\, p_2 \tau \quad \tau > 0 \tag{38}$$

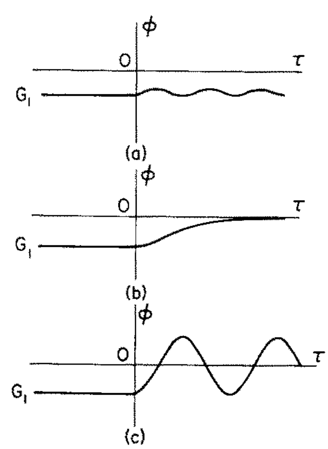

Fig. 5. Nonlinear solutions for $p^2 > 0$: (a) $B < \gamma_3 B_{cr}$, (b) $B = \gamma_3 B_{cr}$, (c) $B > \gamma_3 B_{cr}$ ($\gamma_3 = 3.6742$).

(iii) For $G_1 > -(2/\beta_1)^{1/2}$, or $B > \gamma_3 B_{cr}$, we have

$$\varphi(\tau) = G_1 \operatorname{cn}\{(\delta_3^2 - 1)p_2\tau, \delta_3/[2(\delta_3^2 - 1)]^{1/2}\} \quad \tau > 0 \tag{39}$$

where $\delta_3^2 \equiv \beta_2 G_1^2 > 2$.

The graphs for the solutions for $\varphi(\tau)$ given above are shown in Fig. 5. We shall discuss these solutions in Section 6.

5. OTHER LIMITING CASES

We now consider the case $p^2 = 0$. By eqn (7a) this implies $\lambda^2 - \lambda^4 - D_1 = 0$. Since we assume $1 - \lambda^2 \neq 0$, it follows from (7a) and (7b) that $p^2\beta = 6\lambda^2/h^2(1-\lambda^2)^2$ and eqn (6) becomes

$$\varphi'' + \frac{6\lambda^2}{h^2(1-\lambda^2)^2}\varphi^3 = -BH(-\tau) \tag{40}$$

We apply the condition that φ tends to a static solution of (40), i.e. to some constant as $\tau \to -\infty$. This constant must be a solution of the equation

$$\frac{6\lambda^2}{h^2(1-\lambda^2)^2}\varphi^3 = -B \tag{41}$$

whose only real solution is

$$\varphi = F \equiv -\left[\frac{Bh^2(1-\lambda^2)^2}{6\lambda^2}\right]^{1/3}. \tag{42}$$

By multiplying eqn (40) by φ' and integrating it once for $\tau < 0$, along with the conditions $\varphi(-\infty) = F$ and $\varphi'(-\infty) = 0$, one can establish that $\varphi'(\tau) \equiv 0$ for $\tau < 0$ and hence

$$\varphi(\tau) \equiv F \quad \tau < 0 \tag{43}$$

Similarly, with $\varphi(0) = F$ and $\varphi'(0) = 0$ we can also determine the head wave as

$$\varphi(\tau) = F \operatorname{cn}[\sqrt{(6)}\lambda F\tau/h(1-\lambda^2), 1/2] \quad \tau > 0 \tag{44}$$

We remark that the result in (44) also agree with those given in eqns (18) and (39) when proper limits of the latter are taken.

We now consider the case $\lambda^2 = 1$. Since p^2 is no longer well defined, eqn (6) is not valid. We go back to eqns (5a) and (5b). Assuming that u_0' is finite, we have from (5a)

$$\frac{\varphi^2}{2} = D_1 \tag{45}$$

Thus if φ is not identically zero, D_1 must be positive and $\varphi = 2D_1$. From eqn (5b) we then have

$$u_0'\varphi + \frac{1}{2}\varphi^3 - \lambda^2\varphi = \frac{A}{\rho h c_p^2}H(-\tau) \tag{46}$$

Solving for u_0' we obtain

$$u_0' = \begin{cases} \lambda^2 - D_1 & \tau > 0 \\ \lambda^2 - D_1 + \dfrac{A}{\sqrt{(2D_1)}\rho h c_p^2} & \tau < 0 \end{cases} \tag{47}$$

The travelling load thus generates a discontinuity in u_0'.

Now if $\lambda^2 = 1$ and D_1 also vanishes, then by eqn (5a) $\varphi \equiv 0$. Equation (5b) then yields no solution for u_0' unless $A = 0$. We conclude that if $\lambda^2 = 1$ and $D_1 \leq 0$, the governing eqns(5a) and (5b) yield no solutions in which u_0' is finite.

6. DISCUSSIONS AND CONCLUSIONS

The solutions given in Sections 3 and 4 are separated by $p^2 > 0$ and $p^2 < 0$. The value p^2 as defined in eqn (7a) depends on λ^2 and D_1. Thus p^2 is known if $\lambda^2(\neq 1)$ and D_1 are given.

D_1 was introduced in eqn (5a) as an integration constant. In all the cases that we have considered, φ tends to a constant as $\tau \to -\infty$. Thus u_0' also tends to a constant as $\tau \to -\infty$.

We remark that D_1 cannot be determined unless some further condition is posed. For instance, in[6] D_1 is determined by the further requirement that u_0 be periodic in τ.

One way to determine D_1 is to prescribe the axial force N_x at $\tau = -\infty$. Indeed, we have[1]

$$N_x = \frac{Eh}{1-\upsilon^2}\left(u_0' + \frac{1}{2}\varphi^2\right) \tag{48}$$

Eliminating u_0' from (5a) and (48) and solving for D_1, we have

$$D_1 = (1-\lambda^2)\frac{1-\upsilon^2}{Eh}N_x + \frac{1}{2}\lambda^2\varphi^2 \tag{49}$$

Thus D_1 is determined by knowing the limiting values of N_x and φ as $\tau \to -\infty$.

With λ^2 and D_1 given p^2 is computed from eqn (7a). In particular, setting $\lambda^2 - \lambda^4 - D_1 = 0$ and solving for λ^2, we have

$$\lambda^2 = \lambda_c^2 \equiv \frac{1 \pm \sqrt{(1-4D_1)}}{2} \tag{50}$$

If $D_1 \leq 0$, the only real positive solution for λ_c^2 is obtained by taking the plus sign in (50) above, and we have the case $p^2 > 0$ or the case $p^2 < 0$ depending on whether λ is smaller or greater than

$$\lambda_c \equiv \left[\frac{1+\sqrt{(1-4D_1)}}{2}\right]^{1/2} (>1).$$

If $0 < D_1 \leq 1/4$, there are two real positive solutions for λ_c^2. We have the case $p^2 > 0$ if

$$\lambda_{c_1} \equiv \left[\frac{1-\sqrt{(1-4D_1)}}{2}\right]^{1/2} < \lambda < \left[\frac{1+\sqrt{(1-4D_1)}}{2}\right]^{1/2} \equiv \lambda_{c_2}$$

and the case $p^2 < 0$ if

$$\lambda < \left[\frac{1-\sqrt{(1-4D_1)}}{2}\right]^{1/2}$$

or

$$\lambda > \left[\frac{1+\sqrt{(1-4D_1)}}{2}\right]^{1/2}.$$

If $D_1 > 1/4$, we have case $p^2 < 0$ regardless of the value for λ.

In the linear theory $N_x \equiv 0$, which is obvious from eqn (48). Suppose we also set $N_x \equiv 0$ here. Then $\lambda_c = 1$. We have the subcritical case $p^2 > 0$ if $\lambda < 1 (c < c_p)$ and the supercritical case $p^2 < 0$ if $\lambda > 1 (c > c_p)$. It should also be obvious, by setting $D_1 = 0$ in eqn (49), that the limiting value of N_x as $\tau \to -\infty$ must be related to that of φ by

$$0 = (1-\lambda^2)\frac{1-\upsilon^2}{Eh}N_x + \frac{1}{2}\lambda^2\varphi^2 \tag{51}$$

We now consider the solution obtained in the previous sections. As we observed at the end of Section 3, a general condition that has to be satisfied by the plate response is that as $\tau \to -\infty$, φ

must tend to a static solution of the dynamic equation under a constant load. This is justified in part in Sections 3 and 4 through a perturbation study. In the case of $p^2 > 0$, a nonlinear solution for φ is given in Section 3 which is constant for $\tau < 0$ and oscillates (about $\varphi = 0$) for $\tau > 0$. This solution exists for all values of the load intensity and is the only solution that satisfies the general condition mentioned above. The nonlinear solution may be generated by the linear solution, with the cosine term in the head wave of the linear solution generating the Jacobian elliptic function cn in the head wave of the nonlinear solution. From Fig. 1 and Fig. 2 it is seen that the nonlinear solution φ is very similar to the corresponding linear solution $B\varphi_0$.

In the case $p^2 < 0$, solutions of two different types are obtained in Section 4. The first is that generated by the linear solution and exists only for $B \leq B_{cr}$, i.e. when the load intensity is not too large. This solution, like the corresponding linear solution, is of the exponential type for both $\tau < 0$ and $\tau > 0$ (see Fig. 3 and Fig. 4). Also, this solution satisfies the general condition mentioned above but is not completely determined until the further condition $\varphi \to 0$ as $\tau \to \infty$ is added. A solution of the second type is completely determined by the general condition mentioned above and it is constant for $\tau<0$. It is interesting to observe that the head wave of this solution takes different forms depending on the load intensity. For very large values of B the head wave oscillates about $\varphi = 0$ and is very similar to the solution found for the case $p^2 < 0$ in Section 3 (Fig. 5c). As B decreases, the head wave changes to an exponential type and approaches zero as $\tau \to \infty$ (Fig. 5b). For small values of B, the head wave oscillates about a nonzero static solution of the dynamic equation for $\tau>0$ (Fig. 5c). We also note that the last solution exists even for $B \leq B_{cr}$.

In Sections 5 a nonlinear solution for φ is obtained for the case $p^2 = 0$ and this solution is simply the common limit of the solution for $p^2 > 0$ and of the solution for $p^2 < 0$ for large values of B. This is in fact what we expect since when B is large, the solution for φ will likewise be large so the linear term in eqn (6) becomes negligible when compared with the nonlinear term and the distinction between the cases $p^2 > 0$ and $p^2 < 0$ becomes insignificant. These results are consistent with those obtained in [5] using the perturbation method. The results in [5] predict that for a nonlinear elastic beam under a moving load, the subcritical modes of response ($p^2 > 0$) are extended into the supercritical region ($p^2 < 0$) and there exist more than one type of supercritical modes of response of the beam, provided that the geometrical nonlinearity in the beam is predominant. (See Fig. 1 of [5].)

We have considered the steady-state response of an elastic plate to a moving line load. As we have seen, one of the main difficulties in obtaining the solution for the plate response is connected with the question of how to pose the radiation conditions for nonlinear problems. We have made attempts here to develop such conditions and then construct exact solutions for the dynamic response of the plate.

We remark that the existence of the steady-state plate response to the moving load has been assumed. This assumption implies that if we switch the load on at some finite time, $t = 0$ say, then seek our solution in the coordinates $(x - ct, y, z, t)$ instead of (x, y, z, t), the explicit dependence of the solution on t disappears as t tends to infinity. It should be pointed out that the existence of a steady-state plate response, as well as the rate at which such a steady-state solution is achieved, also depends strongly on the various dampings that we have not included in the plate theory. It is well known that a simple harmonic oscillator under a suddenly applied load never reaches a steady-state solution unless damping is considered, though the latter has little effect on the form of the steady-state solution. We expect an analogous situation to prevail in the plate problem here.

The formulation of the moving load problem as a transient one and then determining the steady-state plate response as the large time limit of the transient solution would, in fact, provide an alternative approach to the problem. Such as approach would not only serve to justify the radiation conditions developed here, but also show how the steady-state plate response is extablished via the transient ones and reveal the effects of various dampings. Some discussions on the relations between transient and steady-state responses of linear elastic beams to moving loads are given by Steele [9].

In the present paper the transient problem is governed by a pair of nonlinear partial differential equations, the exact solutions of which would be extremely difficult. We even encounter considerable difficulty in attempting to obtain numerical solutions by the finite difference method, partly because the solution domain involved is infinite and partly because the presence of the term representing rotatory inertia in the equations makes the finite difference scheme implicit. We do plan, however, to present such numerical solutions in the near future.

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